MATH3210 - SPRING 2024 - SECTION 004

HOMEWORK 9 - *SOLUTIONS*

Problem 1 (30 points)**.** Let $a = d_0 < d_1 < d_2 < \cdots < d_k = b$ be a finite list, $g_i : [d_i, d_{i+1}] \to \mathbb{R}$ be a continuous function for every $i = 1, \ldots, k$, and $f : [a, b] \to \mathbb{R}$ be the function defined by

$$f(x) = g_i(x) \qquad \text{when } x \in [d_i, d_{i+1})$$

and $f(b) = g_k(b)$. Show that f is integrable.

Solution. Fix $\varepsilon > 0$. Since each g_i is continuous it is bounded, and the list is finite, there exists a common bound B such that $|f(x)| \leq B$ for all $x \in [a.b]$. Choose a partition $\mathcal{P}_i$ of $[d_i, d_{i+1}]$ such that $U(g_i, \mathcal{P}_i) - L(g_i, \mathcal{P}_i) < \varepsilon/2k$. Without loss of generality, by adding the element $d_{i+1} - \varepsilon/(4Bk)$ as necessary, we may assume that the last subinterval of $\mathcal{P}_i$ has length at most $\varepsilon/(4Bk)$. Then since f agrees with g_i on $[d_i, d_{i+1}]$ except at d_{i+1}, it follows that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \left[\sum_{i=1}^{k} U(g_i, \mathcal{P}_i) - L(g_{,i}, \mathcal{P}_i)\right] + k \cdot \varepsilon/(4Bk) \cdot (B - (-B)) \leq k \cdot (\varepsilon/2B) + \varepsilon/2 = \varepsilon$$

□

Problem 2 (30 points)**.** Let u and v be continuously differentiable functions on $[a, b]$, and V be an antiderivative of v. Show that

$$\int_a^b uv\, dx = u(b)V(b) - u(a)V(a) - \int_a^b Vu'\, dx.$$

[*Hint*: Apply the fundamental theorems to the fuction $H(y) = \displaystyle\int_a^y uv\, dx$]

Solution. Consider the funciton $H(y) = \displaystyle\int_a^y uv\, dx$. Then since uv is continuous, H is continuously differentiable and $H'(y) = u(y)v(y)$ for all $y \in (a, b)$, and $H(a) = 0$. Let $G(y) = u(y)V(y) - u(a)V(a) - \int_a^y V(x)u'(x)\, dx$. Then $G(a) = u(a)V(a) - u(a)V(a) + \int_a^a V(x)u'(x)\, dx = 0$ and

$$G'(y) = (u'(y)V(y) + u(y)V'(y)) - V(y)u'(y) = u(y)V'(y) = u(y)v(y).$$

Since antiderivatives are unique up to a constant and $G(a) = H(a)$, it follows that $H = G$ as functions on $[a, b]$. The desired equality is exactly $G(b) = H(b)$. □

Problem 3 (40 points)**.** Let $f_n : (a, b) \to \mathbb{R}$ be a sequence of functions on an open interval (a, b). For each, prove or find a counterexample:

(a) If each f_n is bounded, and $f_n \to f$ pointwise, then f is bounded.
(b) If each f_n is bounded, and $f_n \to f$ uniformly, then f is bounded.

Solution. (a) This is *false.* Consider the piecewise functions defined on $(0, 1)$ by

$$f_n(x) = \begin{cases} n^2 x, & 0 < x < 1/n \\ 1/x, & 1/n \leq x \leq 1 \end{cases}.$$

Then each f_n is bounded, the bound given exactly by n. But f_n converges to $1/x$ pointwise, since for each individual x, the function $f_n(x)$ is equal to $1/x$ for $n \geq 1/x$. Crucially, since the bound on n needs to depend on x, f_n does not converge uniformly.

(b) This is *true.* Indeed, if $f_n \to f$ uniformly, then there exists an N such that if $n \geq N$, $|f_n(x) - f(x)| \leq 1$ for all $x \in (a, b)$. Since f_N is bounded, there exists some B such that $|f_N(x)| \leq B$ for all $x \in (a, b)$. Then

$$|f(x)| = |f(x) - f_N(x) + f_N(x)| \leq |f_N(x) - f(x)| + |f_N(x)| \leq 1 + B$$

So $1 + B$ is a bound for f.

$\square$